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By a pass of i units we mean one which, if complete, has an expected gain of i distance units. The available distances are $i = 1, \ldots, k$. Let p_i denote the probability that such a pass is complete, let q_i denote the probability that the pass is incomplete, and suppose the probability that it is intercepted is βq_i for some number β. Then from

$$p_i + q_i + \beta q_i = 1$$

it follows that

$$q_i = \frac{1 - p_i}{1 + \beta}. \tag{1}$$

Let x_i denote the defense resources devoted to stopping a pass of i units. Then p_i is a decreasing function of x_i.

Let C_i denote the offense's probability of winning if a pass of i units is complete, and let I_i be their win probability if the pass is incomplete. Then when the pass is thrown, the offense's probability of winning is

$$f_i(x_i) = p_i(x_i) C_i + \frac{1 - p_i(x_i)}{1 + \beta} I_i. \tag{2}$$

Obviously, the offense will choose an i for which $f_i(x_i)$ is maximized. Therefore, the defense has to choose $x_1, \ldots, x_k$ to minimize the maximum of the $f_i(x_i)$, subject to the constraint that $\sum_i x_i \leq X$, where X represents the total resources available. We can express the defense's optimization problem as

$$\min_{v, x_1, \ldots, x_k} v$$

subject to

$$v \geq f_i(x_i) \quad \text{for all } i, \tag{3}$$

$$\sum_{i=1}^{k} x_i \leq X, \quad \text{and} \tag{4}$$

$$x_i \geq 0 \quad \text{for all } i. \tag{5}$$

The Lagrangean is

$$v + \sum_{i=1}^{k} \theta_i \left[f_i(x_i) - v \right] + \lambda \left[\sum_{i=1}^{k} x_i - X \right] - \mu_1 x_1 - \ldots - \mu_k x_k \tag{6}$$

where λ, the θ_i, and the μ_i are non-negative Lagrange multipliers. The optimality conditions (in addition to the constraints) are

$$1 - \sum_{i=1}^{k} \theta_i = 0, \tag{7}$$

$$\theta_i f_i'(x_i) + \lambda - \mu_i = 0 \quad \text{for all } i, \tag{8}$$

$$\theta_i \left[f_i(x_i) - v \right] = 0 \quad \text{for all } i, \tag{9}$$

$$\lambda \left[\sum_{i=1}^{k} x_i - X \right] = 0, \quad \text{and} \tag{10}$$

$$\mu_i x_i = 0 \quad \text{for all } i. \tag{11}$$

The resource constraint 4 will be binding, so λ will be positive.

Now if $f_i(x_i) \neq v$ for some i, then $\theta_i = 0$ by equation 9. Then, by equation 8, μ_i *cannot* be zero, and hence $x_i = 0$ by equation 11. So for every i, either $f_i(x_i) = v$, or $x_i = 0$. In words, the optimal strategy for the defense is to use its resources in such a way that every pass distance gives the same win probability to the offense—except for those distances that the defense ignores entirely. Let $\bar{v}$ and $\bar{x}_1, \ldots, \bar{x}_k$ denote the solution to the defense's optimization problem. There is actually an easy way to solve for these values: Given any v, let $\bar{x}_i(v)$ be the solution to $v = f_i(x_i)$, if there exists a positive x_i satisfying that equation, and let $\bar{x}_i(v) = 0$ otherwise. Then $\bar{v}$ is the value of v for which $\sum_i \bar{x}_i(v) = X$, and the $\bar{x}_i(\bar{v})$ are the defense's optimal resource allocation.

In equilibrium, the optimal strategy for the offense will, in general, be a randomized strategy, in which a pass of i units will be chosen with probability π_i. In the model it's unnecessary to know the offense's randomization probabilities; we require only $\bar{v}$. However, we can easily determine the π_i by the condition that the $\bar{x}_i$ be optimal against them. Given the π_i, the defense's optimization problem can be alternatively expressed as

$$\min_{x_1,\ldots,x_k} \sum_{i=1}^{k} \pi_i f_i(x_i)$$

subject to $\sum_i x_i - X \leq 0$, and $-x_i \leq 0$ for all i. The Lagrangean is

$$\sum_{i=1}^{k} \pi_i f_i(x_i) + \lambda \left[\sum_{i=1}^{k} x_i - X \right] - \mu_1 x_1 - \ldots - \mu_k x_k \tag{12}$$

where λ and the μ_i are non-negative Lagrange multipliers. We know that the $\bar{x}_i$ are the solution to this optimization problem. Therefore the $\bar{x}_i$ satisfy the optimality conditions, which (in addition to the constraints) are

$$\pi_i f_i'(\bar{x}_i) + \lambda - \mu_i = 0 \quad \text{for all } i, \tag{13}$$

$$\lambda \cdot \left(\sum_{i=1}^{k} \bar{x}_i - X \right) = 0, \quad \text{and} \tag{14}$$

$$\mu_i \bar{x}_i = 0 \quad \text{for all } i. \tag{15}$$

Equations 13, 14, and 15 can now be regarded as equations for the π_i. If $\bar{x}_i = 0$, we can simply set $\pi_i = 0$. Otherwise, $\mu_i = 0$, and equation 13 together with the requirement that $\sum_i \pi_i = 1$ determines the π_i.