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We have to show that there exist unique numbers $V_1, \ldots, V_n$, with $0 \leq V_i \leq 1$ for all i, satisfying the equations

$$V_i = 1 - \sum_j c_{ij} V_j \tag{1}$$

for $i = 1, \ldots, n$. These equations can be written as

$$V = \iota - CV \tag{2}$$

where ι is an n-vector of ones, and the matrix C has i, j element c_{ij}.

Let $\Gamma = \{x \in \mathbf{R}^n : 0 \leq x_i \leq 1 \text{ for all } i\}$, and define $f : \Gamma \to \mathbf{R}^n$ by $f(V) = \iota - CV$. Then the goal can be equivalently stated as showing that there is a unique $V \in \Gamma$ such that $f(V) = V$. Actually, since the s_i are strictly positive, it follows from

$$s_i + \sum_j c_{ij} = 1 \tag{3}$$

that $\sum_j c_{ij} < 1$ for all i. Hence

$$0 \leq \sum_j c_{ij} V_j \leq \sum_j c_{ij} < 1 \tag{4}$$

for each i and every $V \in \Gamma$, from which it follows that $f(\Gamma) \subset \Gamma$. It therefore suffices to show that f is a contraction mapping.

Let

$$\gamma = \max_i \sum_j c_{ij} < 1. \tag{5}$$

For the norm on $\mathbf{R}^n$ it's convenient to use

$$|x| = \max_i |x_i|. \tag{6}$$

Then for any $x \in \mathbf{R}^n$,

$$|Cx| = \max_i |\sum_j c_{ij} x_j| \leq \max_i \sum_j c_{ij} |x_j| \leq |x| \max_i \sum_j c_{ij} = \gamma |x|. \tag{7}$$

Hence for $V, W \in \Gamma$,

$$|f(V) - f(W)| = |C(V - W)| \leq \gamma |V - W|. \tag{8}$$