The set of possible outcomes of a regular-season game is $\Omega = \{\text{win, lose, tie}\}$. We assume we have preferences over the probability distributions on Ω. We seek conditions under which those preferences can be represented by a value function V defined on the set of distributions. By this we mean that if x and y are distributions on Ω, then x is strictly preferred to y if and only if $V(x) > V(y)$. In addition, we want the value function to commute with the operation of taking mixtures of distributions. By this we mean that if $x_1, \ldots, x_n$ are distributions on Ω and $\alpha_1, \ldots, \alpha_n$ are non-negative numbers summing to 1, then

$$V\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \sum_{i=1}^{n} \alpha_i V(x_i). \tag{1}$$

This property allows us to use the value function in the dynamic program. To see why, suppose that from the current state $\mathcal{S}_0$, the transition to state $\mathcal{S}_i$ occurs with probability α_i. Suppose further that at state $\mathcal{S}_i$, the probability distribution on Ω is x_i. Then at $\mathcal{S}_0$ the probability distribution on Ω is $\sum_i \alpha_i x_i$. If equation 1 holds, the value at the current state is the weighted average of the values at the states to which a transition could occur, weighted by the probabilities of the transition.

We will use lower-case Roman letters to denote probability distributions, and use Greek letters to denote numbers in the interval $[0, 1]$.

Let $\mathcal{P}$ denote the set of all probability distributions on Ω. For $x, y \in \mathcal{P}$, write $x \succeq y$ to mean that x is at least as good as y. If $x \succeq y$ and $y \succeq x$, then we are indifferent between x and y, and we write $x \sim y$. If $x \succeq y$ but not $y \succeq x$, then x is strictly preferred to y, and we write $x \succ y$. Notice that $x \sim y$ does not imply that x and y are identical, just that they are equally good. We write $x = y$ if x and y are identical.

We assume that for every $x, y \in \mathcal{P}$, either $x \succeq y$ or $y \succeq x$ (or both). We also assume that if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

Let w denote the distribution for which the probability of winning is 1, and let ℓ denote the distribution for which the probability of losing is 1. Then $w \succ \ell$, and $w \succeq x \succeq \ell$ for every x.

The existence of a value function that meets our requirements follows easily from the following axioms[1]:

- A1 (Continuity): If $x \succeq z \succeq y$, there exists α such that

$$z \sim \alpha x + (1-\alpha)y.$$

- A2 (Monotonicity): If $x \succ y$ and $\alpha > \beta$, then

$$\alpha x + (1-\alpha)y \succ \beta x + (1-\beta)y.$$

- A3 (Substitution): If $x \sim y$, then $\alpha x + (1-\alpha)z \sim \alpha y + (1-\alpha)z$ for every α and every z.

We define the value function $V : \mathcal{P} \to \mathbf{R}$ as follows. For $x \in \mathcal{P}$, we let $V(x) \in [0,1]$ be the number such that

$$x \sim V(x)w + (1-V(x))\ell.$$

$V(x)$ exists by A1 and is unique by A2. Axiom A2 also implies that $V(x) > V(y)$ if and only if $x \succ y$, so V is a representation of the preferences. To see that V commutes with mixtures, suppose $x_1, \ldots, x_n$ are distributions, and $\alpha_1, \ldots, \alpha_n$ are non-negative numbers summing to 1. We apply A3 repeatedly to the distribution $\sum_i \alpha_i x_i$, successively replacing each x_i by

$$V(x_i)w + (1-V(x_i))\ell.$$

This yields

$$\begin{aligned} \sum_{i=1}^n \alpha_i x_i &\sim \sum_{i=1}^n \alpha_i [V(x_i)w + (1-V(x_i))\ell] \\ &= \left(\sum_{i=1}^n \alpha_i V(x_i)\right) w + \left(1 - \sum_{i=1}^n \alpha_i V(x_i)\right) \ell. \end{aligned}$$

From the definition of V, it follows that

$$V\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=1}^n \alpha_i V(x_i).$$

[1]A value function exists under weaker assumptions, but the proof isn't so simple. For an example see I.N. Herstein and J. Milnor, "An Axiomatic Approach to Measurable Utility," *Econometrica*, 1953.