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Assume that there are n possible plays that the offense could run. Associated with each defensive strategy is a vector $(q_1, \ldots, q_n)$, where q_i is the probability that the ith play results in a first down. Further assume that the set of all such vectors forms a convex subset of $\mathbf{R}^n$ whose relevant boundary consists of those vectors for which $h(q_1, \ldots, q_n) \geq 0$, where h is some real-valued function. The assumption of convexity is harmless because the defense can use randomized strategies. We assume that $h_i(q_1, \ldots, q_n) \geq 0$, where h_i denotes partial derivative with respect to the ith argument. When $h_i(q_1, \ldots, q_n)$ is strictly positive, the defense can achieve a reduction in q_j for $j \neq i$ by increasing q_i.

Let p_i be the offense's win probability conditional on running play i and making a first down. Let s be the offense's win probability if they fail to make a first down. We assume $p_i > s$ for all i.

Given a defense strategy characterized by $(q_1, \ldots, q_n)$, the optimal strategy for the offense is to run a play for which their win probability

$$q_i p_i + (1 - q_i)s \tag{1}$$

is largest. It follows that the defense should minimize the maximum value (over i) of expression (1). Formally, the decision problem for the defense is

$$\min_{v, q_1, \ldots, q_n} v$$

subject to

$$v \geq q_i p_i + (1 - q_i)s \quad \text{for all } i, \tag{2}$$

$$h(q_1, \ldots, q_n) \geq 0. \tag{3}$$

The Lagrangean is

$$v + \sum_{i=1}^{n} \theta_i \left[q_i p_i + (1 - q_i)s - v \right] - \mu h(q_1, \ldots, q_n) \tag{4}$$

where the θ_i and μ are non-negative Lagrange multipliers. The optimality conditions (in addition to the constraints) are

$$1 - \sum_{i=1}^{n} \theta_i = 0, \tag{5}$$

$$\theta_i(p_i - s) - \mu h_i(q_1, \ldots, q_n) = 0 \quad \text{for all } i, \tag{6}$$

$$\theta_i \left[q_i p_i + (1 - q_i)s - v \right] = 0 \quad \text{for all } i, \tag{7}$$

$$-\mu h(q_1, \ldots, q_n) = 0. \tag{8}$$

Clearly constraint (3) will be binding, so μ will be positive. There are two cases, depending on whether $h_i(q_1, \ldots, q_n)$ is positive or zero. If $h_i(q_1, \ldots, q_n)$ is positive, then equation (6) implies that θ_i cannot be zero, and so by equation (7),

$$q_i p_i + (1 - q_i)s = v. \tag{9}$$

But if $h_i(q_1, \ldots, q_n)$ is zero, then θ_i must be zero, and we can have

$$q_i p_i + (1 - q_i)s < v. \tag{10}$$

Denote the optimal value of q_i by $\bar{q}_i$. Equations (5) and (6) can now be used to determine the θ_i. It turns out that these Lagrange multipliers are in fact the randomization probabilities used by the offense in the Nash equilibrium. To see this, let π_i be the probability that the offense chooses play i, and note that the defense's problem of minimizing the offense's win probability can be alternatively expressed as

$$\min_{q_1, \ldots, q_n} \sum_{i=1}^{n} \pi_i \left[q_i p_i + (1 - q_i)s \right]$$

subject to

$$h(q_1, \ldots, q_n) \geq 0. \tag{11}$$

The Lagrangean is

$$\sum_{i=1}^{n} \pi_i \left[q_i p_i + (1 - q_i)s \right] - \delta h(q_1, \ldots, q_n) \tag{12}$$

where δ is a non-negative Lagrange multiplier. We know that the $\bar{q}_i$ are the solution to this optimization problem. Therefore the $\bar{q}_i$ satisfy the optimality conditions, which (in addition to the constraints) are

$$\pi_i(p_i - s) - \delta h_i(\bar{q}_1, \ldots, \bar{q}_n) = 0 \quad \text{for all } i, \tag{13}$$

$$-\delta h(\bar{q}_1, \ldots, \bar{q}_n) = 0. \tag{14}$$

Equations (13) and (14) can now be regarded as equations for the π_i. Equation (13) and the requirement that the randomization probabilities sum to one are identical to the equations (6) and (5) that determine the θ_i, as claimed.

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