Consider two situations that are identical except that one of them is third down and the other is fourth down. Let v_i be the offense's conditional win probability given outcome i if it was third down, and let w_i be the offense's conditional win probability given outcome i if it was fourth down. We assume that there is an advantage to gaining more yards. Therefore, there are outcomes j and k, both of which pick up the first down, such that $v_k > v_j$. Let m be an outcome that does not pick up the first down; we assume that $w_m < v_m$.

The sufficient condition for third down and fourth down to be equivalent is that there exist a and b, with $b > 0$, such that $w_i = a + bv_i$ for all i. Suppose that such a and b exist. Then in particular

$$w_j = a + bv_j \tag{1}$$

$$w_k = a + bv_k \tag{2}$$

$$w_m = a + bv_m. \tag{3}$$

However, since outcomes j and k both pick up the first down, $w_j = v_j$ and $w_k = v_k$. Equations (1) and (2) therefore imply $a = 0$ and $b = 1$, which in turn imply that $w_m = v_m$, contradicting the assumption that $w_m < v_m$. Therefore, no such a and b can exist.